Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, Second Semester
Semestral Examination - 2013-2014
Graph Theory

Time: 3 Hours May 7, 2014 Instructor: C.R.E. Raja

Total Marks: 50
Section I: Answer any four, each question is worth 6 marks

1. Prove that a graph of order n and size greater than $\lfloor \frac{n^2}{4} \rfloor$ has a triangle. Is the result true if the size of the graph is $\lfloor \frac{n^2}{4} \rfloor$? Justify your answer.
2. Prove that a regular bipartite graph with $\delta(G) \geq 2$ has no bridge.
3. Prove that a connected graph has an Euler circuit if and only if each vertex has even degree.
4. Let G be a bipartite graph with bipartition (X, Y). Suppose $d(x) \geq 1$ for all $x \in X$ and for $x \in X$ and $y \in Y$ with $xy \in E(G)$, we have $d(x) > d(y)$. Prove that G has a complete matching from X to Y.
5. How many 1-factors does a tree have? Justify your answer.
6. Define fundamental cycle vectors and cocycle vectors and prove that they are orthogonal.

Section II: Answer any two, each question is worth 13 marks

1. Let G be a tree and $l(G)$ be the number of vertices of degree one.

   (a) Prove that every edge is a bridge and G is maximal acyclic. (Marks 4)

   (b) Prove that G has an isolated vertex or $l(G) \geq 2$ (Marks 3)

   (c) If $l(G) = 2$, is there a path containing all the vertices? Justify your answer.
2. (a) Let G be a graph with vertices $1, \cdots, n$ and A be the adjacency matrix of G. What is the graphical interpretation of the (i, j)-th entry of A^k for any $k \geq 1$. Justify your answer.

   (b) Let G be a bipartite graph with bipartition (V_1, V_2). Prove that there is a complete matching from V_1 to V_2 if and only if $|\Gamma(S)| \geq |S|$ for all $S \subset V_1$.
3. If $\delta(G) = \frac{|V|}{2}$ or G is a k-regular bipartite graph with $k \geq 1$, prove that G has a 1-factor.